Comment on "Dynamics of some neural network models with delay"

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Based upon numerical evidence, Ruan *et al.* [J. Ruan, L. Li, and W. Lin, Phys. Rev. E **63**, 051906 (2001)] suggest that the delay differential equation $dx/dt(t)=-x(t)+A\tanh[x(t)]+B\tanh[x(t-\tau)]$ may display chaotic dynamics. As mentioned by Pakdaman and Malta [IEEE Trans. Neural Netw. **9**, 231 (1998)], this equation presents a monotonic delayed feedback, so that it satisfies a Poincaré-Bendixson-like theorem, ruling out the existence of complex aperiodic dynamics.

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In Ref. [1], the dynamical threshold neuron model with a single delay was transformed into the following delay differential equation:

$$\frac{dx}{dt}(t)=-x(t)+A\tanh[x(t)]+B\tanh[x(t-\tau)], \qquad (1)$$

where τ is the delay.

Gopalsamy and Leung [1] proved that for $A>0$, $B<0$, and $(A-B)<1$, Eq. (1) is globally asymptotically stable. Pakdaman and Malta [2] discussed the dynamics of Eq. (1) in other parameter regions and determined the parameter ranges where most trajectories stabilize at equilibria, and those where the delay leads to stable periodic oscillations. This description exhausted all possible asymptotic dynamics for Eq. (1).

Ruan *et al.* [3] have further investigated the dynamics of the system (1) in the A,B parameter space, as a function of the delay τ. Their analytical results are obtained using a very nice method that involves a Lyapunov functional, and are in total agreement with our results in Ref. [2]. However, based on numerical simulations, they suggested that the system (1) may exhibit chaotic dynamics. The existence of chaos was ruled out in Ref. [2] because for monotonic delayed feedback, the Poincaré-Bendixson theorem [4] applies, implying that the asymptotic dynamics of Eq. (1) cannot be more complex than those of a two-dimensional system. This precludes the existence of chaos in system (1).

In Fig. 1, we display the time series, and the corresponding projection $x(t)$ versus $x(t-\tau)$, for the three cases of presumed chaotic dynamic presented in Ref. [3]. In the absence of any information regarding the numerical method used in [3], we did the calculation with two numerical methods, namely, the first order explicit Euler scheme, and the 4th order Runge-Kutta scheme adapted to Eq. (1). Both numerical methods produced the same results. As we can see from the left column in Fig. 1, in all cases the oscillation is periodic. Also, as predicted by the Poincaré-Bendixson theorem, the projection of the periodic solution onto the $x(t)-x(t-\tau)$ plane forms a closed loop, similar to a planar limit cycle, in the sense that the projected trajectory does not cross itself (right column in Fig. 1).

In addition, given that a single neuron with monotonic feedback cannot exhibit chaotic dynamics, this will also be true in the special case of two noninteracting neurons with monotone feedback for which Ruan *et al.* [3] presented numerical evidence of chaotic dynamics. In conclusion, the aperiodic chaoticlike dynamics reported by Ruan *et al.* [3] are not supported by the analytical results on delay differential equations with monotonic feedback. It is likely that these behaviors are specific to the numerical method used by Ruan *et al.* [3] to approximate the solution of the delay differential equation. It should be remarked that the Gear three-step method used by Malta and Teles [5] is not applicable to Eq. (1) due the presence of the instantaneous nonlinear term $A\tanh[x(t)]$ (instantaneous feedback loop). Investigations of the Runge-Kutta method applied to delay differential equations can be found in Ref. [6], for instance.

Finally, any scalar delay differential equation with a single delayed monotonic feedback loop [like Eq. (1)] constitute a good test case for any numerical method: if the solutions obtained numerically exhibit a dynamical behavior that is not in agreement with the description provided by Pakdaman and Malta [2], then either the computer code has errors or the numerical method is not suitable.

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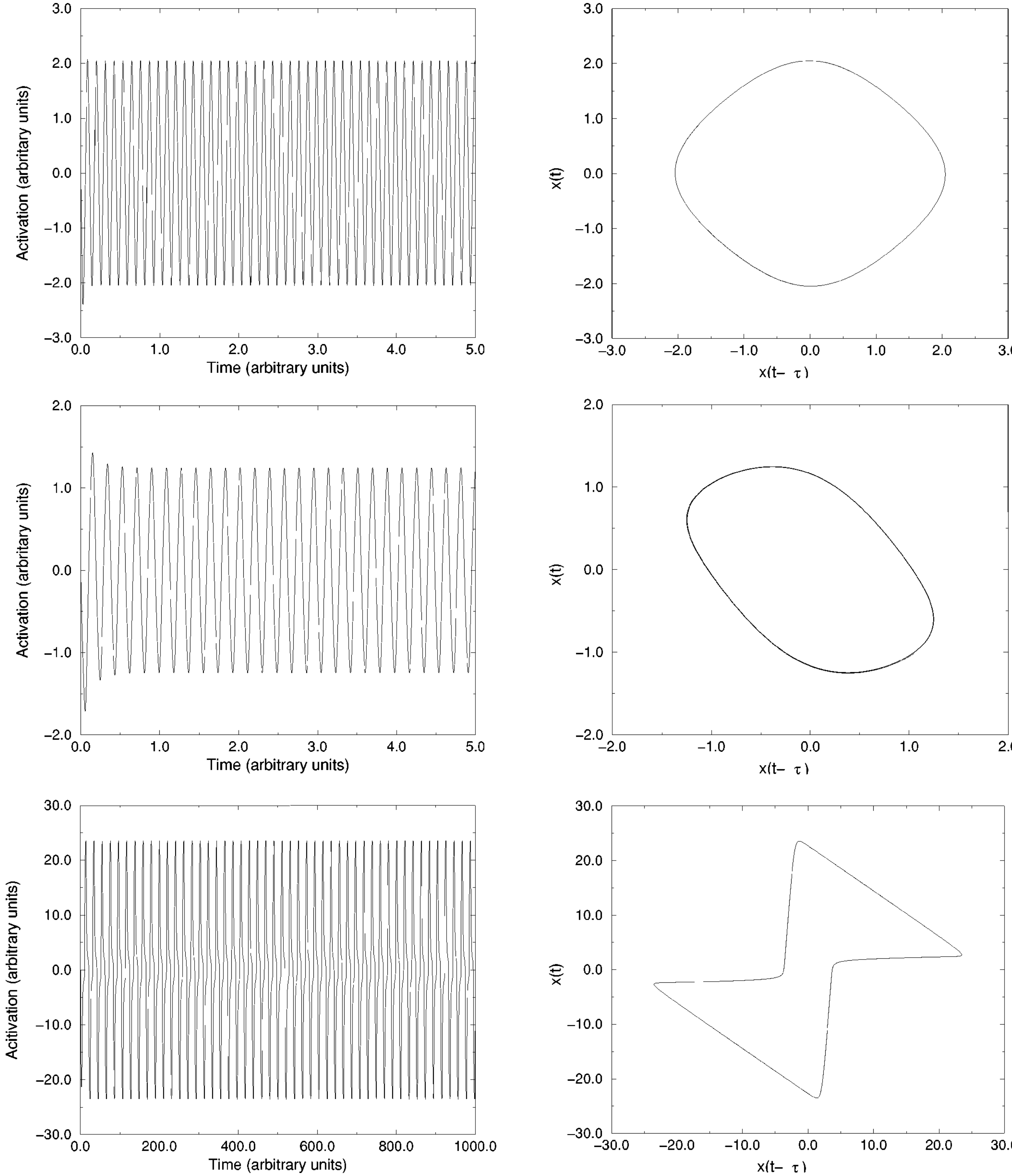

FIG. 1. Left column shows the time series of activation $x(t)$ against time; right column shows activation $x(t)$ versus $x(t-\tau)$ (abscissas and ordinates in arbitrary units). Top panel shows $A=1.0$, $B=-100.0$, with $\tau=0.028$; middle panel shows $A=-20.0$, $B=-50.0$, with $\tau=0.06$; bottom panel shows $A=13.5373$, $B=-11.4627$, with $\tau=3.90$. The calculations were done with time step $\tau/1000$, using the same initial conditions of Ruan *et al.* [3].

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